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Relativistic scattering amplitude in the Pöschl-Teller double ring-shaped Coulomb potential

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HIGHLIGHTS

- The continuous energy states have been obtained for the radial part of the Dirac equation.
- The relativistic scattering amplitude have calculated for spin $\frac{1}{2}$ particles.
- The Pöschl-Teller double ring-shaped Coulomb potential have been used for solution of the relativistic equation.

ABSTRACT

In this research, we obtain the exact solution to the Dirac equation with the Pöschl-Teller double ring-shaped Coulomb (PTDRSC) potential for any spin-orbit quantum number κ. The relativistic scattering amplitude for spin $\frac{1}{2}$ particles in the field of this potential has been studied. The wave functions are being expressed in terms of the hyper-geometric series of the continuous states on the $\frac{k}{2\pi}$ scale. In addition, a formula for the phase shifts has also been found.

KEYWORDS

Dirac equation
Scattering state
Phase shifts
Pöschl-Teller Potential
Ring-shaped Potential

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1 Introduction

The solution to relativistic equations plays an essential role in many aspects of modern physics. In particular, the Dirac equation is the most frequently used wave equation in the description of particle dynamics in relativistic quantum mechanics and in many fields of physics such as nuclear and high-energy physics as well as chemistry. In recent years, there has been an increase trend in searching for analytic solution to the Dirac equation; For example, see (Jia et al., 2009; Aydoğdu and Sever, 2010; Zhang and Huang-Fu, 2012; Eshghi and Ikhdair, 2014b,c; Moghadam et al., 2013; Eshghi and Mehraban, 2012a; Xue-Ao et al., 2005; Cheng and Dai, 2007; Eshghi and Mehraban, 2012b; Eshghi and Hamzavi, 2012; Eshghi et al., 2017; Zarrinkamar et al., 2010).

On the other hand, scattering theory is worried with the external dynamics of unbounded particles flowing from and to unlimited, and having continuous energy spectra (Taylor, 2006; Joachain, 1975). Conversely, the interior dynamics involves the bound states, which form discrete energy spectra and generate quasiperiodic time evolvements. In this regard, there is a large variety of scattering systems in areas such as for example particle, nuclear, atomic, molecular, chemical, and mesoscopic physics, photonics, phononics, surface science, gas kinetics, geophysics, astrophysics, etc. In fact, scattering systems are characterized by infinity movement before and after collision between particles or with a barrier.

By permitting the formulation of quantum mechanics in systems of infinite spatial expansion, scattering theory paves the best way to the explanation of transport properties for open systems in contact with particle and heat reservoirs (Nazarov and Blanter, 2009; Ferry and Good-

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nick, 1999; Datta, 1995; Imry, 1997). If the reservoirs have various temperatures and chemical potentials, such open systems are out of equilibrium and they generate thermodynamic entropy. Since Landauers pioneering work (Landauer, 1957), advances have demonstrated how the transport and thermodynamic properties may be precisely formulated on the foundation of scattering theory (Büttiker et al., 1985; Levitov et al., 1996; Blanter and Büttiker, 2000; Tasaki, 2001; Tasaki and Takahashi, 2006; Bruneau et al., 2013; Sâad and Pillet, 2014; Gaspard, 2015b,a).

Therefore, the scattering problems, in the presence of an external potential field, have become highly interesting topics in relativistic and non-relativistic quantum mechanics. Its well known that the scattering of a relativistic particle in the field of a potential can be treated exactly by finding the continuum solutions of the Dirac equation.

Also, there has been continuous growing interest in studying the scattering states solution within the framework of non-relativistic and relativistic quantum mechanics for central and non-central potentials alike (Yazarloo et al., 2015; Edet et al., 2021; Arda, 2017; Arda et al., 2010; Eshghi and Abdi, 2013; Dong and Lozada-Cassou, 2004; Chang-Yuan et al., 2013; Ochiai and Nakazato, 2018; Wang et al., 2020; Sakhnovich, 2019; Tesfahun, 2020; Motohashi and Noda, 2021).

In the present work, we intend to solve the Dirac equation with the PTDRSC potential (Fa-Lin and Chang-Yuan, 2010) for its scattering states. The physical form of this potential (Fa-Lin and Chang-Yuan, 2010) is given in spherical coordinates as

$$V(r,\theta,\phi)=-\frac{\delta}{r}+\frac{1}{r^2}\Big[\frac{B}{\sin^2\theta}+\frac{A(A-1)}{\cos^2\theta}\Big]+\frac{1}{r^2\sin^2\theta}\Big[\frac{\alpha^2D(D-1)}{\sin^2\alpha\phi}+\frac{\alpha^2C(C-1)}{\cos^2\alpha\phi}\Big] \tag{1}$$

where $A, C, D > 1$, $\delta > 0$, $B > 0$, and $\alpha = 1, 2, 3, ...$ are real positive parameters. However, some authors such as Sun et al, You et al, and Chen et al have used the double ring-shaped potential for solution of the Schrödinger equation by using the various techniques (Sun et al., 2015a,b; You et al., 2018; Chen et al., 2016). But, Maghsoodi et al. (Maghsoodi et al., 2013) have solved the Dirac equation for the above potential with the Nkiforov-Uvarov (NU) method. Here, we attempt to study the scattering states of the Dirac equation for the above potential and will discuss some of its analytical properties. This article is organized as follows: In Section 2, we intend to solve the Dirac equation with the PTDRSC potential for any spin-orbit quantum number. In Section 3, we obtain the continuous energy states along with the wave functions for the radial parts of the Dirac equation. Finally, we end with our discussion and conclusions in Section 4.

2 The Dirac Equation

We begin by presenting the Dirac Hamiltonian (in natural units $\hbar = c = 1$) is (Lisboa et al., 2004; Akcay, 2009):

$$H=\vec{\alpha}.\vec{p}+\beta\big(M+S(\vec{r})\big)+V(\vec{r}) \tag{2}$$

$$\Big[\vec{\alpha}.\vec{p}+\beta\big(M+S(\vec{r})\big)+V(\vec{r})\Big]\Psi(\vec{r}))=E\Psi(\vec{r}) \tag{3}$$

where E denotes the energy. In Pauli-Dirac representation, because of the appearance of 4×4 matrices in the Dirac equation, the wave function must be a four-component vector. It is necessary to classify the upper two and lower two components of the Dirac wave function as two-component spinors (Dyall and Fægri Jr, 2007). We write

$$\Psi(\vec{r})=\begin{pmatrix}\varphi^L(r)\\ \chi^S(r)\end{pmatrix}\equiv\begin{pmatrix}\varphi(r)\\ \chi(r)\end{pmatrix} \tag{4}$$

where $\varphi^L(r)$ and $\chi^S(r)$ are termed the *large* and *small* components of the wave function, we get

$$\vec{\sigma}.\vec{p}\chi(\vec{r})=\big[E-V(\vec{r})-M-S(\vec{r})\big]\varphi(\vec{r}) \tag{5}$$

$$\vec{\sigma}.\vec{p}\varphi(\vec{r})=\big[E-V(\vec{r})+M+S(\vec{r})\big]\chi(\vec{r}). \tag{6}$$

In the case when scalar potential is equal to the vector potential, the above equations turn out to become

$$\vec{\sigma}.\vec{p}\chi(\vec{r})=\big[E-M-2V(\vec{r})\big]\varphi(\vec{r}) \tag{7}$$

$$\chi(\vec{r})=\frac{\vec{\sigma}.\vec{p}}{E+M}\varphi(\vec{r}) \tag{8}$$

Further, after substituting Eq. (8) into Eq. (7), we can obtain a Schrödinger-like equation for the upper component:

$$\big[p^2+2(E+M)V(\vec{r})\big]\varphi(\vec{r})=\big[E^2-M^2\big]\varphi(\vec{r}) \tag{9}$$

and afterward plugging the potential (1) into Eq. (9), we can obtain

$$\Bigg[-\nabla^2+2(E+M)\times\left(\begin{array}{c}-\frac{\delta}{r}+\frac{1}{r^2}\Big[\frac{B}{\sin^2\theta}+\frac{A(A-1)}{\cos^2\theta}\Big]\\ \frac{1}{r^2\sin^2\theta}\Big[\frac{\alpha^2D(D-1)}{\sin^2\alpha\phi}+\frac{\alpha^2C(C-1)}{\cos^2\alpha\phi}\Big]\end{array}\right)\Bigg]\varphi(r,\theta,\phi)=\big[E^2-M^2\big]\varphi(r,\theta,\phi) \tag{10}$$

In order to make separation of variables in a spherical coordinates for the desired spherical potential, we resort to substitute the following ansatz of the wave function

$$\varphi_{nlm}(r,\theta,\phi)=\frac{g_{nlm}(r)}{r}\frac{H_l(\theta)}{(\sin\theta)^{1/2}}\Phi_m(\phi) \tag{11}$$

into Eq. (10) which leads to the three set of second-order differential equations:

$$\frac{d^2g(r)}{dr^2}+\Bigg\{\Big(\frac{2(M+E)\delta}{r}+\frac{\frac{1}{4}-\ell^2}{r^2}\Big)-M^2+E^2\Bigg\}g(r)=0 \tag{12}$$

$$\frac{d^2H(\theta)}{d\theta^2}+\Bigg\{\Big(\frac{\frac{1}{4}-2(M+E)-m^2}{\sin^2\theta}-\frac{2(M+E)A(A-1)}{\cos^2\theta}\Big)+\ell^2\Bigg\}H(\theta)=0 \tag{13}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2}+\left\{\left(\frac{-2(M+E)\alpha^2D(D-1)}{\sin^2\alpha\phi}-\frac{2(M+E)\alpha^2C(C-1)}{\cos^2\alpha\phi}\right)+m^2\right\}\Phi(\phi)=0 \tag{14}$$

where $\ell=0,1,2,\dots$ and $m=0,\pm1,\pm2,\dots$ are separation constants.

2.1 Polar and azimuthal solutions

For this case, by choosing

$$\begin{aligned}&-\frac{\frac{1}{4}-2(E+M)-m^2}{\zeta^2}=\chi(\chi-1)\\&-\frac{2(E+M)A(A-1)}{\zeta^2}=\lambda(\lambda-1)\\&\ell^2=2E+\zeta^2\end{aligned} \tag{15}$$

we can write Eq. (13) as

$$-\frac{1}{2}\frac{d^2H(q)}{dq^2}+\frac{\zeta^2}{2}\left\{\frac{\chi(\chi-1)}{\sin^2(\zeta q)}+\frac{\lambda(\lambda-1)}{\cos^2(\zeta q)}-\left(E+\frac{1}{2}\zeta^2\right)\right\}H(q)=0 \tag{16}$$

Equation (47) is obviously a standard one-dimensional form of the Schrödinger equation with a generalized Pöchl-Teller effective potential which admits an exact solution of the form

$$E_{nr}=\frac{\zeta^2}{2}(\chi+\lambda+2n_r)^2 \tag{17}$$

$$\begin{aligned}H(q)=&C\sin^{\chi}(\zeta q)\cos^{\lambda}(\zeta q)\\&\times{}_2F_1(-n_r,\chi+\lambda+n_r,\chi+\frac{1}{2};\sin^2(\zeta q))\end{aligned} \tag{18}$$

with $\chi,\lambda>1$, $H(0)=0$, and $H(\frac{\pi}{2\zeta})=0$, as reported by Salem and Montemayor, Eq. (4.7) in (Salem and Montemayor, 1993). However, for $\chi=0,1$ effective potential (16) collapses into

$$V_{\text{eff}}(q(r))=\frac{\zeta^2}{2}\frac{\lambda(\lambda-1)}{\cos^2(\zeta q)} \tag{19}$$

which admits an exact solution

$$\begin{aligned}&E_{n_r}=2\,\zeta^2\big(\frac{\lambda}{2}+n_r\big)^2-\frac{\zeta^2}{2}\\&H(q)=A\cos^{\lambda}(\zeta q)\\&\qquad\times{}_2F_1\Big(-n_r,\lambda+n_r,\frac{1}{2};\sin^2(\zeta q)\Big)\end{aligned} \tag{20}$$

We can also obtain exact solution of the Eq. (13) by using the NU method (Nikiforov and Uvarov, 1988). The analytical exact solution of Eq. (13) has been given by (Maghsoodi et al., 2013) by NU method. To obtain a solution of Eq. (14) and to avoid repetition in our solution, If we substitute $-2(E+M)\alpha^2D(D-1)=\chi(\chi-1)$, $-2(E+M)\alpha^2C(C-1)=\lambda(\lambda-1)$, and $m^2=E+\frac{1}{2}\zeta^2$, then Eq. (14) turns to Eq. (16). Using the similar procedure like the ones in above the present subsection, eigenvalues and eigenfunctions of the Eq. (14) can be easily obtained. The analytical exact solution of Eq. (14) has also been given in Ref. (Maghsoodi et al., 2013) by NU method.

2.2 Solution of the radial Dirac equation

To solution of Eq. (12), we can simply use the Laplace transform (Spiegel, 1965) that the analytical exact solution of Eq. (12) has also been given in Ref. (Eshghi and Ikhdair, 2014a) by this method. We use the answers obtained as Eq. (24) into Ref. (Eshghi and Ikhdair, 2014a), and write the answer of the radial part as follows:

$$-\frac{(E+M)\delta}{\sqrt{E^2-M^2}}-\sqrt{\frac{1}{2}-\ell^2}+\frac{1}{2}=n \tag{21}$$

where $n=0,1,2,\dots$ is principle quantum numbers, respectively. Given that in many-body systems such as nuclear and stellar materials or electronic systems, the study of scattering states and thermodynamic discussions is one of the most basic tasks (Typel, 2013; Berakdar, 2001).

Here, we can apply this idea of studying thermodynamic properties for the radial limit state (non-relativistic state), and consider our system as a canonical ensemble and calculate the partition function. Due to overlapping topics, we are postponing this part of the calculation for future works.

3 Scattering States of the Radial Dirac Equation

If the energy is positive, then $k=\sqrt{E_{n\kappa}^2-M^2}$ is called the wave number associated to the electron whenever it moves asymptotically for the origin, the center of the force field. Eq. (12) can be written as

$$\frac{d^2g(r)}{dr^2}+\left[k^2+\frac{2(M+E_{n\kappa})\delta}{r}-\frac{(\ell-\frac{1}{2})(\ell+\frac{1}{2})}{r^2}\right]g(r)=0 \tag{22}$$

The boundary conditions for Eq. (22) on the boundaries $g(0)=0$ and $g(\infty)$ are to be finite values. Owing to the asymptotic behavior of the radial wave functions of the continuous states as $r\to\infty$, we need to take the wave functions in the form

$$g(r)=A(kr)^{\ell+1/2}\exp(ikr)f(r) \tag{23}$$

On substituting the wave function (23) into Eq. (22), we have

$$\frac{d^2f}{dr^2}+(2\ell+1+2ikr)\frac{df}{dr}+\left[2ik(\ell+\frac{1}{2})+2(M+E_{n\kappa})\delta\right]f=0 \tag{24}$$

Changing to a new variable $z=-2ikr$, Eq. (24) can be simplified as

$$\frac{d^2f}{dz^2}+(2\ell+1-z)\frac{df}{dz}+\left[\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\right]f=0 \tag{25}$$

whose analytical solutions as $r \to \infty$ are the confluent hypergeometric functions (Berakdar, 2001; Wang and Guo, 1979)

$$f(r) = F\Big(\ell + \frac{1}{2} - \frac{i(M+E_{n\kappa})\delta}{k}, 2\ell+1, -2ikr\Big) \tag{26}$$

Thus, the radial wave function of the scattering states are expressed

$$\begin{aligned} g_{k\ell}(r) &= A_{k\ell}(kr)^{\ell+1/2}\exp(ikr) \\ &\times F\Big(\ell + \frac{1}{2} - \frac{i(M+E_{n\kappa})\delta}{k}, 2\ell+1, -2ikr\Big) \end{aligned} \tag{27}$$

We now study asymptotic form of the above expression for large r to calculate the normalization constant $A_{k\ell}$ of radial wave functions and the phase shifts δ'_ℓ. Further, the asymptotic expression of the confluent hypergeometric functions when $|z| \to \infty$ is given by (Berakdar, 2001; Wang and Guo, 1979):

$$F(\eta,\gamma,z) \to \frac{\Gamma(y)}{\Gamma(\eta)}\mathrm{e}^{z} z^{\eta-\gamma} + \frac{\Gamma(y)}{\Gamma(\gamma-\eta)}\mathrm{e}^{\pm i\pi\eta} z^{-\eta} \tag{28}$$

The upper sign in the second term applies for $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$ and the lower sign in the second term applies for $-\frac{3\pi}{2} < \arg z < -\frac{\pi}{2}$, and the symbol Γ denotes the Gamma function. When $z = -2ikr = |z|\mathrm{e}^{-i\pi/2}$, Eq. (28) is then re-expressed as

$$\begin{aligned} F(\eta,\gamma,z) \to & \frac{\Gamma(y)}{\Gamma(\eta)}\mathrm{e}^{z} z^{\eta-\gamma}\mathrm{e}^{-i\pi(\eta-\gamma)/2} \\ &+ \frac{\Gamma(y)}{\Gamma(\gamma-\eta)}\mathrm{e}^{-i\pi\eta/2} z^{-\eta} \end{aligned} \tag{29}$$

from which, we have

$$\begin{aligned} &F\Big(\ell + \frac{1}{2} - \frac{i(M+E_{n\kappa})\delta}{k}, 2\ell+1, -2ikr\Big) \\ &\xrightarrow{r\to\infty} \Gamma(2\ell+1)(2kr)^{-\left(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\right)}\mathrm{e}^{-2ikr} \\ &\times \frac{\exp\Big(i\pi\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\big)/2\Big)}{\Gamma\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\big)} \\ &+ \frac{\Gamma(2\ell+1)(2kr)^{-\left(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\right)}}{\Gamma\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\big)} \\ &\times \exp\Big(i\pi\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\big)/2\Big) \end{aligned} \tag{30}$$

If we can write

$$\begin{aligned} &\Gamma\Big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\Big) \\ &= \Big|\Gamma\Big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\Big)\Big|\exp(i\delta_\ell) \end{aligned} \tag{31}$$

then

$$\begin{aligned} &\Gamma\Big(\ell+\frac{1}{2}+\frac{i(M+E_{n\kappa})\delta}{k}\Big) \\ &= \Big|\Gamma\Big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\Big)\Big|\exp(-i\delta_\ell) \end{aligned} \tag{32}$$

where δ_ℓ is a real number. Equation (30) then becomes

$$\begin{aligned} &F\Big(\ell + \frac{1}{2} - \frac{i(M+E_{n\kappa})\delta}{k}, 2\ell+1, -2ikr\Big) \\ &\approx \frac{\Gamma(2\ell+1)(2ikr)^{-\left(\ell+\frac{1}{2}+\frac{i(M+E_{n\kappa})\delta}{k}\right)}}{\Big|\Gamma\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\big)\Big|} \\ &+ \frac{\Gamma(2\ell+1)\,\mathrm{e}^{-2ikr}(2kr)^{-\left(\ell+\frac{1}{2}+\frac{i(M+E_{n\kappa})\delta}{k}\right)}}{\Big|\Gamma\big(\ell+\frac{1}{2}+\frac{i(M+E_{n\kappa})\delta}{k}\big)\Big|} \end{aligned} \tag{33}$$

then

$$\begin{aligned} &F\Big(\ell + \frac{1}{2} - \frac{i(M+E_{n\kappa})\delta}{k}, 2\ell+1, -2ikr\Big) \\ &= \frac{\Gamma(2\ell+1)\exp(-ikr)\exp\big(-\frac{\pi(M+E_{n\kappa})\delta}{2k}\big)}{\Big|\Gamma\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\big)\Big|(2kr)^{\ell+1/2}} \\ &\times \Bigg[(-i)^{-\ell-1/2}\exp\Big(-i\Big(kr+\delta_\ell-\frac{\ell\pi}{2} \\ &+ \frac{(M+E_{n\kappa})\delta\ln(2kr)}{k}\Big)\Big) \\ &- i^{-\ell-1/2}\exp\Big(i\Big(kr+\delta_\ell-\frac{\ell\pi}{2} \\ &+ \frac{(M+E_{n\kappa})\delta\ln(2kr)}{k}\Big)\Big)\Bigg] \end{aligned} \tag{34}$$

By $r \to \infty$, we have

$$\begin{aligned} &F\Big(\ell + \frac{1}{2} - \frac{i(M+E_{n\kappa})\delta}{k}, 2\ell+1, -2ikr\Big) \\ &\xrightarrow{r\to\infty} \frac{\Gamma(2\ell+1)\exp(-ikr)\exp\big(-\frac{\pi(M+E_{n\kappa})\delta}{2k}\big)}{\Big|\Gamma\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\big)\Big|(2kr)^{\ell+1/2}} \\ &\times \Big[i\exp\Big(-i\Big(kr+\delta_\ell-\frac{\ell\pi}{2}+\frac{(M+E_{n\kappa})\delta\ln(2kr)}{k}\Big)\Big) \\ &- i\exp\Big(i\Big(kr+\delta_\ell-\frac{\ell\pi}{2}+\frac{(M+E_{n\kappa})\delta\ln(2kr)}{k}\Big)\Big)\Big] \end{aligned} \tag{35}$$

Substituting Eq. (35) into Eq. (27) leads to

$$\begin{aligned} &g_{k\ell}(r) \xrightarrow{r\to\infty} \frac{2A_{k\ell}(kr)\Gamma(2\ell+1)\exp\big(-\frac{\pi(M+E_{n\kappa})\delta}{2k}\big)}{\Big|\Gamma\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}\big)\Big|} \\ &\times \sin\Big(kr+\delta_\ell-\frac{\ell\pi}{2}+\frac{\pi}{4}+\frac{(M+E_{n\kappa})\delta\ln(2kr)}{k}\Big) \end{aligned} \tag{36}$$

In terms of the following asymptotic behavior

$$\begin{aligned} g_{k\ell}(r) \xrightarrow{r\to\infty} & 2\sin\Big(kr+\delta_\ell-\frac{\ell\pi}{2}+\frac{\pi}{4} \\ &+ \frac{(M+E_{n\kappa})\delta\ln(2kr)}{k}\Big) \end{aligned} \tag{37}$$

It is given in Ref. (Landau and Lifshitz, 2013; Zeng, 2000) that the radial wave functions of the continuous states for the Coulomb potential are normalized on the "$\frac{k}{2\pi}$ scale". Because the new model potential is a short distance potential in the PTDRSC Potential, so it has no influence on asymptotic expression of the wave function for large r.

It is useful to note that, considering the asymptotic behavior of the wave function, the scattering amplitudes are also valid if we take into account the relativity (Chen et al., 2004; Schiff, 1995). In other words, the asymptotic expression of the PTDRSC potential is identical to that of the Coulomb potential when $r \to \infty$, i.e.

$$g_{k\ell}(r) \xrightarrow{r\to\infty} 2\sin\Big(kr + \delta'_\ell - \frac{\ell\pi}{2} + \frac{\pi}{4} + \frac{(M+E_{n\kappa})\delta\ln(2kr)}{k}\Big) \tag{38}$$

The wave functions of the continuous states for the PT-DRSC potential are normalized on the "$\frac{k}{2\pi}$ scale", too. Here δ'_ℓ represents the phase shifts. If we compare Eq. (38) with Eq. (30), we may obtain the normalization constant of the continuous states as

$$A_{k\ell}(r) = \frac{2^{\ell+1}\Big|\Gamma\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta)}{k}\big)\Big|}{\Gamma(2\ell+1)} \times \exp\Big(\frac{\pi(M+E_{n\kappa})\delta)}{2k}\Big) \tag{39}$$

and the phase shifts δ'_ℓ for a short ranged interaction. It is the additional ℓ'-independent phase shift, $-(M+E_{n\kappa})\delta\ln(2kr)/k$, that distinguishes the Coulomb-like solution from that for a short ranged potential and can be calculated explicitly:

$$\begin{aligned}\delta'_\ell &= \delta_\ell + \pi(\ell'-\ell+\frac{1}{2})/2 \\ &= \arg\Gamma\Big(\ell+\frac{1}{2}-i(M+E_{n\kappa})\delta/k\Big) \\ &+ \pi(\ell'-\ell+\frac{1}{2})/2\end{aligned} \tag{40}$$

where ℓ' depends to E and ζ similar to ℓ.

Substituting Eq. (39) into (27), the normalized wave functions of the continuous states on the "$\frac{k}{2\pi}$ scale" are:

$$\begin{aligned}g_{k\ell}(r) = &\frac{2^{\ell+1}\Big|\Gamma\big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta)}{k}\big)\Big|}{\Gamma(2\ell+1)} \\ &\times \exp\Big(\frac{\pi(M+E_{n\kappa})\delta)}{2k}\Big)\exp(ikr)(kr)^{\ell+1/2} \\ &\times F\Big(\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k}, 2\ell+1, -2ikr\Big)\end{aligned} \tag{41}$$

where $k = \sqrt{E_{n\kappa}^2 - M^2}$.

Before concluding this section, let us study the properties of the scattering amplitude. As we know, once the phase shifts are obtained, we can study the scattering amplitude and the differential cross section. For the sake of simplicity, following (Lin, 1999) we can obtain the scattering amplitude as

$$f(\theta) = -\frac{i}{\sqrt{2\pi k}}\sum_\ell \Big[\exp(2i\delta_\ell - 1)\Big]e^{im\theta} \tag{42}$$

Due to $\sum_\ell e^{im\theta} = 2\pi\delta(\theta)$, we have for $\theta \neq 0$

$$f(\theta) = -\frac{i}{\sqrt{2\pi k}}\sum_i \Big[\exp(2i\delta_\ell)\Big]e^{im\theta} \tag{43}$$

from which we may calculate the cross section

$$\sigma(\theta) = |f(\theta)|^2 = \Big| -\frac{i}{\sqrt{2\pi k}}\sum_i \big[\exp(2i\delta_\ell)\big]e^{im\theta}\Big|^2 \tag{44}$$

It should be noted that it is very difficult to obtain an analytical expression for Eq. (37). Nevertheless, Dong and Lozada-Cassou (Dong and Lozada-Cassou, 2004) have obtained cross section in the special case as

$$\sigma(\theta) = |f(\theta)|^2 = \Big| -i\frac{\Gamma(1/2-i\alpha)\,e^{i\alpha\ln\left(\sin^2(\theta/2)\right)}}{\sqrt{2k}\,\Gamma(i\alpha)\sin(\theta/2)}\Big| \tag{45}$$

by

$$\Gamma(iy)\Gamma(-iy) = \big|\Gamma(iy)\big|^2 = \frac{\pi}{y\sinh(\pi y)}$$

and

$$\Gamma(\frac{1}{2}+iy)\Gamma(\frac{1}{2}-iy) = \big|\Gamma(\frac{1}{2}+iy)\big|^2 = \frac{\pi}{y\cosh(\pi y)}$$

we have

$$\sigma(\theta) = \frac{\alpha\tanh(\pi\alpha)}{2k\sin^2(\theta/2)}$$

For more information about scattering, see Appendix A.

3.1 Analytical properties of the scattering amplitude

Here we shall discuss the analytical properties of the scattering amplitude in the entire complex k plane by regarding the scattering amplitude as the function of the phase shifts. To this end, from Eq. (35), we need discuss analytical properties of $\Gamma\big(\ell+\frac{1}{2}-i(M+E_{n\kappa})\delta/k\big)$. The Gamma function, $\Gamma(z)$, has simple poles at $z = 0, -1, -2, \ldots$. To see this we can use to write

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \ldots \tag{46}$$

Clearly, Gamma function, $\Gamma(z)$, has a pole at $z = 0$ with residue, at $z = -1$ with residue, at $z = -2$ with residue, etc. Also $\Gamma(z)$ is never zero in the complex plane. Namely, the first order poles of $\Gamma\big(\ell+\frac{1}{2}-i(M+E_{n\kappa})\delta/k\big)$ is situated at

$$\ell+\frac{1}{2}-\frac{i(M+E_{n\kappa})\delta}{k} = 0, -1, -2, ... = -n_r \tag{47}$$

where $n_r = 0, 1, 2, \ldots$. At these poles, the corresponding energy levels are given by Eq. (21).

4 Discussion and Conclusions

In this work, we have investigated the solution of the Dirac equation for particles with spin $\frac{1}{2}$ in the PTDRSC potential. The continuous energy states of the Dirac equation with this potential have been presented for any spin-orbit quantum number κ. The wave functions have been expressed in terms of the hyper-geometric series of the continuous states on the $\frac{k}{2\pi}$ scale. Also, formula of the phase shifts was calculated. We recovered the nonrelativistic solutions in the limiting case. We also presented some of the analytical scattering amplitude.

Appendix A: Review to Scattering Cross-Section

Once the differential scattering cross-section is known, the total scattering cross-section σ_{tot} and the first transport scattering cross section σ_{tr} may be calculated by $\sigma_{tot} = \int(\mathrm{d}\sigma/\mathrm{d}\Omega)\mathrm{d}\Omega$ and $\sigma_{tr} = \int(1-\cos\theta)(\mathrm{d}\sigma/\mathrm{d}\Omega)\mathrm{d}\Omega$. We can also obtain the ratio Ξ between the transport and the total scattering cross section as $\Xi = \sigma_{tr}/\sigma_{tot}$. A quantity which is very important for describing the scattering processes in the interaction of electrons with the matter, is the already mentioned transport cross section σ_{tr}. It is now easy to calculate the probability of scattering into angular rang from 0 to θ, that is given by $P(\theta) = (2\pi/\sigma_{tot})\int_0^\infty(\mathrm{d}\sigma/\mathrm{d}\Omega)\sin\vartheta\,\mathrm{d}\vartheta$. Other useful quantities that can be given by simple closed formulas are the probability of forward scattering as, and the probability of backscattering as. The knowledge of the forward and of the backscattering probabilities allows us to calculate the backscattering coefficient $r(E_0)$ (Dapor, 2004; Dapor et al., 2000; Vicanek and Urbassek, 1991). For low atomic number elements and for some oxides, the differential scattering cross section can be approximated by the function as

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